Appl. Math. J. Chinese Univ.
2008, 23(3): 351-358

3-restricted connectivity of graphs with given girth

GUO Li-tao   MENG Ji-xiang

Abstract. Let $G = (V, E)$ be a connected graph. $X \subset V(G)$ is a vertex set. X is a 3-restricted cut of G, if $G-X$ is not connected and every component of $G-X$ has at least three vertices. The 3-restricted connectivity $\kappa_3(G)$ (in short κ_3) of G is the cardinality of a minimum 3-restricted cut of G. X is called κ_3-cut, if $|X| = \kappa_3$. A graph G is κ_3-connected, if a 3-restricted cut exists. Let G be a graph girth $g \geq 4$, $\xi_3(G)$ is $\min\{d(x) + d(y) + d(z) - 4 : xyz$ is a 2-path of $G\}$. It will be shown that $\kappa_3(G) = \xi_3(G)$ under the condition of girth.

§1 Introduction

A network is often modelled by a graph $G = (V, E)$ with the vertices representing nodes such as processors or stations, and the edges representing links between the nodes. Throughout this paper, we assume the graphs considered are simple.

Let $G = (V, E)$ be a connected graph. For a vertex $v \in V$, $N(v)$ is the set of all vertices adjacent to v. The degree of a vertex v is $d(v) = |N(v)|$. $\delta = \delta(G)$ is the minimum degree of G. If $u, v \in V$, $d(u, v)$ denotes the length of a shortest (u, v)-path. If $X, Y \subset V$, $d(X, Y) = \min\{d(x, y) :$ for any $x \in X$ and any $y \in Y\}$ denotes the distance between X and Y. $v \in V, r \geq 0$ is an integer, $N_r(v) = \{w \in V : d(w, v) = r\}$, $N_1(v) = N(v)$. For $X \subset V$, $N_r(X) = \{w \in V : d(w, X) = r\}$ where $d(w, X) = d(\{w\}, X)$, $N_1(X) = N(X)$. $N[v] = N(v) \cup \{v\}, N[X] = N(X) \cup X$. $G[X]$ is the subgraph induced by X. We denote the diameter and girth by D and g, respectively, and write $G - v$ for $G - \{v\}$. A path is called k-path, if its length of edges is k.

An edge set S is called a k-restricted edge cut of G, if $G - S$ is not connected and every connected component of $G-S$ has at least k vertices. The cardinality of a minimum k-restricted edge cut is the k-restricted edge connectivity of G, denoted by $\lambda_k(G)$. As for the recent studies in this aspect, we can see [2-9].

Received: 2008-03-11
MR Subject Classification: 05C
Keywords: 3-restricted cut, 3-restricted connectivity, girth
Digital Object Identifier(DOI): 10.1007/s11766-008-1908-z
Supported by the National Natural Science Foundation of China (10671165) and Specialized Research Fund for the Doctoral Program of Higher Education of China (20050755001)

A vertex set X is a k-restricted cut of G, if $G-X$ is not connected and every component of $G-X$ has at least k vertices. The k-restricted connectivity $\kappa_k(G)$ (in short κ_k) of G, is the cardinality of a minimum k-restricted cut of G. X is called κ_k-cut, if $|X|=\kappa_k$. Not all connected graphs have κ_k-cuts, for example $K_{1,n-1}$ has no κ_k-cuts, $k\geq 2$. A graph G is κ_k-connected, if a κ_k-cut exists.

Let G be a connected graph with girth $g\geq 4$, $\xi_3(G)$ (in short ξ_3) is $\min\{d(x)+d(y)+d(z)-4 : xyz$ is a 2-path of $G\}$. But the relation between $\kappa_3(G)$ and $\xi_3(G)$ is not certain. The following figures illustrate it.

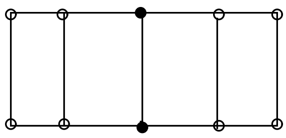

Fig. (a) $\kappa_3 < \xi_3$

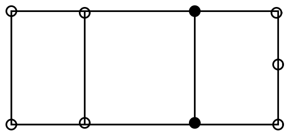

Fig. (b) $\kappa_3 = \xi_3$

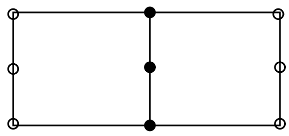

Fig. (c) $\kappa_3 > \xi_3$

In [1], the authors have studied the κ_2-connected graphs. We will study the relation between $\kappa_3(G)$ and $\xi_3(G)$ of κ_3-connected graphs with given girth in the next part.

§2 Main results

Lemma 2.1. Let G be a connected graph with girth $g\geq 6$, and minimum degree $\delta\geq 3$. Then G is κ_3-connected and $\kappa_3(G)\leq\xi_3(G)$, if one of the following assertions holds :
(1) $g\geq 7$ or $\delta\geq 4$;
(2) there exists a 2-path $u_0u_1u_2$ with $d(u_0)+d(u_1)+d(u_2)-4=\xi_3(G)$ such that every cycle $u_0u_1u_2u_3u_4u_5u_0$ (if any) satisfies $d(u_4)\geq 4$.

Proof. (1) Let xyz be any 2-path of G. It is not difficult to see that for all $w\in V-N[\{x,y,z\}]$ there exist at least two vertices u,v in $N(w)$ such that $\{u,v\}\cap N(\{x,y,z\})=\varnothing$ whenever $g\geq 7$ or $\delta\geq 4$. Hence $N(\{x,y,z\})$ is a 3-restricted cut, following that G is κ_3-connected and $\kappa_3(G)\leq|N(\{x,y,z\})|$ for any 2-path of G. So $\kappa_3(G)\leq\xi_3(G)$.

(2) Let $u_0u_1u_2$ be a 2-path satisfying the hypothesis of the lemma. Suppose that for any

$$z\in V-N[\{u_0,u_1,u_2\}],$$

if $N(z)\subseteq N(\{u_0,u_1,u_2\})$, then there is a cycle less than g. If there is only a vertex $u\in N(z), u\overline{\in}N(\{u_0,u_1,u_2\})$, as $g\geq 6$, so $N(u_0)\cap N(z)=\{u_5\}, N(u_2)\cap N(z)=\{u_3\}, d(z)=3$. We have a cycle $u_0u_1u_2u_3zu_5u_0$ with $d(z)=3$, against the hypothesis. Thus G is κ_3-connected. Observe that $\kappa_3(G)\leq|N(\{u_0,u_1,u_2\})|=\xi_3(G)$.

Let $G=(V,E)$ be a connected graph, $X\subset V$, $v\in V\setminus X$ and $u\in N(v)$. We introduce the sets
$X_u^+(v)=\{z\in N(v)-u : d(z,X)=d(v,X)+1\}$,
$X_u^=(v)=\{z\in N(v)-u : d(z,X)=d(v,X)\}$,
$X_u^-(v)=\{z\in N(v)-u : d(z,X)=d(v,X)-1\}$.
Clearly, $X_u^+(v), X_u^=(v), X_u^-(v)$ form a partition of $N(v)-u$, and $|X_u^+(v)|+|X_u^=(v)|+|X_u^-(v)|=d(v)-1$. If $d(v)\geq 2$, $u,w\in N(v)$, then
$X_{uw}^+(v)=\{z\in N(v)-\{u,w\} : d(z,X)=d(v,X)+1\}$,

$X^{=}_{uw}(v) = \{z \in N(v) - \{u,w\} : d(z,X) = d(v,X)\}$,
$X^{-}_{uw}(v) = \{z \in N(v) - \{u,w\} : d(z,X) = d(v,X) - 1\}$. Clearly, $X^{+}_{uw}(v), X^{=}_{uw}(v), X^{-}_{uw}(v)$ form a partition of $N(v) - \{u,w\}$, and $|X^{+}_{uw}(v)| + |X^{=}_{uw}(v)| + |X^{-}_{uw}(v)| = d(v) - 2$.

Let G be a κ_3-connected graph with girth $g \geq 4$ and minimum degree $\delta \geq 3$. If $X \subset V$ is a κ_3-cut, then for each connected component C of $G - X$, we assume $\mu = \max\{d(u,X) : u \in V(C)\}$. If $2 \leq \mu \leq \lceil (g-4)/2 \rceil - 1$, then we have the following lemmas.

Lemma 2.2. Let G be a κ_3-connected graph with girth $g \geq 4$, and minimum degree $\delta \geq 3$. Let $X \subset V$ be a κ_3-cut. For each connected component C of $G - X$, if there exists a 2-path xyz in C such that $d(x,X) = d(y,X) = d(z,X) = \mu$, then $\kappa_3(G) \geq \xi_3(G)$.

Proof. In this case, $X^{+}_{y}(x) = X^{+}_{y}(z) = X^{+}_{xz}(y) = \varnothing$ and $|N_\mu(X^{=}_{y}(x)) \cap X| \geq |X^{=}_{y}(x)|$. Otherwise, there are two vertices x_1, x_2 in $X^{=}_{y}(x)$ such that $N_\mu(x_1) \cap N_\mu(x_2) \cap X = \{x_3\}$. Then the cycle formed by the (x_1,x_3)-path, (x_3,x_2)-path and edges xx_1, xx_2 has length $2\mu+2 < g$, a contradiction. Similarly, $|N_\mu(X^{=}_{y}(z)) \cap X| \geq |X^{=}_{y}(z)|, |N_\mu(X^{=}_{xz}(y)) \cap X| \geq |X^{=}_{xz}(y)|; |N_\mu(x) \cap X| \geq |X^{-}_{y}(x)|, |N_\mu(z) \cap X| \geq |X^{-}_{y}(z)|, |N_\mu(y) \cap X| \geq |X^{-}_{xz}(y)|$. Because $N_\mu(X^{=}_{y}(x)) \cap X, N_\mu(X^{=}_{y}(z)) \cap X, N_\mu(X^{=}_{xz}(y)) \cap X, N_\mu(x) \cap X, N_\mu(z) \cap X$ and $N_\mu(y) \cap X$ are pairwise disjoint. Otherwise, say, there are $v_1 \in X^{=}_{y}(x), v_2 \in X^{=}_{y}(z), v_3 \in N_\mu(X^{=}_{y}(x)) \cap X$ such that both v_1 and v_2 are at distance μ to v_3.Then there is a cycle going through $\{x,y,z,v_2,v_3,v_1\}$ of length at most $2\mu + 4 \leq 2\lceil (g-4)/2 \rceil + 2 \leq g - 1$, which is impossible.

Therefore, we have

$$\begin{aligned}
\kappa_3(G) = |X| &\geq |N_\mu(X^{=}_{y}(x)) \cap X| + |N_\mu(X^{=}_{y}(z)) \cap X| + |N_\mu(X^{=}_{xz}(y)) \cap X| \\
&\quad + |N_\mu(x) \cap X| + |N_\mu(z) \cap X| + |N_\mu(y) \cap X| \\
&\geq |X^{=}_{y}(x)| + |X^{=}_{y}(z)| + |X^{=}_{xz}(y)| + |X^{-}_{y}(x)| + |X^{-}_{y}(z)| + |X^{-}_{xz}(y)| \\
&= d(x) + d(y) + d(z) - 4 \geq \xi_3(G).
\end{aligned}$$

Lemma 2.3. Let G be a κ_3-connected graph with girth $g \geq 4$, and minimum degree $\delta \geq 3$. Let $X \subset V$ be a κ_3-cut. For each connected component C of $G - X$, if there exists a 2-path xyz in C such that $d(x,X) = d(y,X) = \mu, d(z,X) = \mu - 1$, then $\kappa_3(G) \geq \xi_3(G)$.

Proof. In this case, $X^{+}_{y}(x) = X^{+}_{xz}(y) = \varnothing$, and it is analogous to Lemma 2.2, we get $|N_\mu(X^{=}_{xz}(y)) \cap X| \geq |X^{=}_{xz}(y)|, |N_\mu(y) \cap X| \geq |X^{-}_{xz}(y)|, |N_\mu(X^{=}_{y}(x)) \cap X| \geq |X^{=}_{y}(x)|, |N_\mu(x) \cap X| \geq |X^{-}_{y}(x)|, |N_\mu(X^{+}_{y}(z)) \cap X| \geq |X^{+}_{y}(z)|, |N_{\mu-1}(X^{=}_{y}(z)) \cap X| \geq |X^{=}_{y}(z)|, |N_{\mu-1}(z) \cap X| \geq |X^{-}_{y}(z)|$.

Similarly to Lemma 2.2, $N_\mu(X^{=}_{xz}(y)) \cap X, N_\mu(y) \cap X, N_\mu(X^{=}_{y}(x)) \cap X, N_\mu(x) \cap X, N_\mu(X^{+}_{y}(z)) \cap X, N_{\mu-1}(X^{=}_{y}(z)) \cap X$ and $N_{\mu-1}(z) \cap X$ are pairwise disjoint, hence we can deduce $\kappa_3(G) \geq \xi_3(G)$.

As Lemma 2.2 and Lemma 2.3, we can obtain the following similar lemmas.

Lemma 2.4. Let G be a κ_3-connected graph with girth $g \geq 4$, and minimum degree $\delta \geq 3$. Let $X \subset V$ be a κ_3-cut. For each connected component C of $G - X$, if there exists a 2-path xyz in C such that $d(x,X) = d(z,X) = \mu, d(y,X) = \mu - 1$, then $\kappa_3(G) \geq \xi_3(G)$.

Lemma 2.5. Let G be a κ_3-connected graph with girth $g \geq 4$, and minimum degree $\delta \geq 3$. Let $X \subset V$ be a κ_3-cut. For each connected component C of $G - X$, if there exists a 2-path xyz in C such that $d(x,X) = \mu, d(y,X) = d(z,X) = \mu - 1$, then $\kappa_3(G) \geq \xi_3(G)$.

Lemma 2.6. Let G be a κ_3-connected graph with girth $g \geq 4$, and minimum degree $\delta \geq 3$.

Let $X \subset V$ be a κ_3-cut. For each connected component C of $G-X$, if there exists a 2-path xyz in C such that $d(x,X)=\mu, d(y,X)=\mu-1, d(z,X)=\mu-2$, then $\kappa_3(G)\geq \xi_3(G)$.

Lemma 2.7. Let G be a κ_3-connected graph with girth $g\geq 4$, and minimum degree $\delta\geq 3$. Let $X\subset V$ be a κ_3-cut. For each connected component C of $G-X$, if every vertex y in C with $d(y,X)=\mu$ is such that each 2-path xyz in C satisfies $d(y,X)=\mu, d(x,X)=d(z,X)=\mu-1$, then $\kappa_3(G)\geq \xi_3(G)$.

Proof. It is analogous to Lemma 2.2, we have $|N_{\mu-1}(N(y)-\{x,z\})\cap X|\geq |N(y)-\{x,z\})|, |N_{\mu-1}(X_y^{=}(z))\cap X|\geq |X_y^{=}(z)|, |N_{\mu-1}(z)\cap X|\geq |X_y^{-}(z)|, |N_{\mu-1}(N(X_y^{+}(z))-z)\cap X|\geq |X_y^{+}(z)|, |N_{\mu-1}(N(X_y^{+}(x))-x)\cap X|\geq |X_y^{+}(x)|, |N_{\mu-1}(X_y^{=}(x))\cap X|\geq |X_y^{=}(x)|, |N_{\mu-1}(x)\cap X|\geq |X_y^{-}(x)|$. And $N_{\mu-1}(N(y)-\{x,z\})\cap X, N_{\mu-1}(X_y^{=}(z))\cap X, N_{\mu-1}(z)\cap X, N_{\mu-1}(N(X_y^{+}(z))-z)\cap X$, $N_{\mu-1}(X_y^{=}(x))\cap X, N_{\mu-1}(x)\cap X$ are pairwise disjoint.

Therefore the required result follows.

Lemma 2.8. Let G be a κ_3-connected graph with girth $g\geq 4$, and minimum degree $\delta\geq 3$. Let $X\subset V$ be a κ_3-cut. If $\kappa_3(G)<\xi_3(G)$, then for each connected component C of $G-X$ there exists some vertex $u\in V(C)$ such that $d(u,X)\geq \lceil (g-4)/2\rceil$.

Proof. For $g=4,5,6$ the result is immediate. So suppose that $g\geq 7$ and let C be any component of $G-X$. Let us denote $\mu=\max\{d(u,X): u\in V(C)\}$. Therefore we can assume that $\mu\geq 2$.

We reason by contradiction, so assume that $2\leq \mu\leq \lceil (g-4)/2\rceil -1$.

Case 1. There exists a 2-path xyz in C such that

Subcase 1.1. $d(x,X)=d(y,X)=d(z,X)=\mu$.

Subcase 1.2. $d(x,X)=d(y,X)=\mu, d(z,X)=\mu-1$.

Subcase 1.3. $d(x,X)=d(z,X)=\mu, d(y,X)=\mu-1$.

Subcase 1.4. $d(x,X)=\mu, d(y,X)=d(z,X)=\mu-1$.

Subcase 1.5. $d(x,X)=\mu, d(y,X)=\mu-1, d(z,X)=\mu-2$.

Case 2. Every vertex y in C with $d(y,X)=\mu$ is such that each 2-path xyz in C satisfies $d(y,X)=\mu, d(x,X)=d(z,X)=\mu-1$.

From Lemmas 2.2–2.7, the result is immediate.

Let G be a κ_3-connected graph with $\kappa_3(G)<\xi_3(G)$. X is a κ_3-cut, for any connected component C of $G-X$, by Lemma 2.8 the following set is nonempty:

$$C_X=\{u\in V(C): d(u,X)\geq \lceil (g-4)/2\rceil\}.$$

Lemma 2.9. Let G be a κ_3-connected graph with even girth $g\geq 6$, minimum degree $\delta\geq 3$. X is a κ_3-cut. Assume that there exists a connected component C of $G-X$ such that $\max\{d(u,X): u\in V(C)\}=(g-4)/2$. Then, if $\kappa_3(G)<\xi_3(G)$, the following assertions hold :

(1) Every vertex in $G[C_X]$ lies on a 2-path in $G[C_X]$.

(2) If $u\in C_X$, then $|N(u)\cap C_X|\geq 2$.

(3) Any vertex $u\in C_X$ lies on a cycle of length g.

(4) There exists a vertex $u\in C_X$ such that $|N_{(g-4)/2}(u)\cap X|=1$.

Proof. (1) When $g=6$, any $u\in C$, $d(u,X)=1$. For X is a κ_3-cut and C is connected, it is done. Hence assume $g\geq 8$. We reason by contradiction that there is a vertex $u\in C_X$, u does not lie on a 2-path.

Case 1. u is an isolated vertex in $G[C_X]$, that is $N(u) \cap C_X = \varnothing$.

In this case, $N(u) \subset C$ since $d(u, X) = (g-4)/2$, and $|N_{(g-4)/2}(u) \cap X| \geq d(u)$ $(g \geq 8, N(u) \cap C_X = \varnothing)$. Take any $u_1, u_2 \in N(u)$, then $d(u_1, X) = d(u_2, X) = (g-6)/2$. It is analogous to Lemma 2.7. We get a contradiction.

Case 2. u lies on an isolated edge in $G[C_X]$.

Take an isolated edge uv in $G[C_X]$, there is a vertex $w \in N(v) \cap V(C)$ such that $d(w, X) = (g-6)/2$. It is analogous to Lemma 2.3.

(2) It is equivalent to showing that $X_v^{=}(u) \neq \varnothing$ and $X_v^{=}(w) \neq \varnothing$ for any 2-path uvw in $G[C_X]$. Suppose $X_v^{=}(u) = \varnothing$. If $g = 6$, then for any $x \in V(C), d(x, X) = 1, N(C) \subseteq X$. If $|V(C)| = 3$, we have $\kappa_3(G) = |X| \geq d(u) + d(v) + d(w) - 4 \geq \xi_3(G)$, a contradiction. So $|V(C)| \geq 4$.

We can take a 2-path uvw in C_X such that $X_v^{=}(u) = \varnothing$. Observe that $X_v^{-}(u) \subseteq X, X_v^{-}(z) \subseteq X, X_{uw}^{-}(v) \subseteq X$ and $X_v^{-}(u), X_v^{-}(z), X_{uw}^{-}(v), N(X_v^{=}(w)) \cap X, N(X_{uw}^{=}(v)) \cap X$ are pairwise disjoint, $|N(X_v^{=}(w)) \cap X| \geq |X_v^{=}(w)|, |N(X_{uw}^{=}(v)) \cap X| \geq |X_{uw}^{=}(v)|$. We deduce $\kappa_3(G) = |X| \geq \xi_3(G)$, which is a contradiction.

Then assume that $g \geq 8$, $d(u, X) \geq 2, N(u) \cap C \neq \varnothing$. Take $x \in N(u) \cap C$ such that $d(x, X) = (g-6)/2$, xuv is a 2-path such that $d(u, X) = d(v, X) = (g-4)/2, X_u^{+}(v) = \varnothing, X_v^{=}(u) = X_{vx}^{=}(u) = \varnothing$ and $X_{vx}^{+}(u) = \varnothing$. It is analogous to Lemma 2.3, we can deduce that $\kappa_3(G) \geq \xi_3(G)$, a contradiction.

(3) By contradiction that a vertex $u \in C_X$ does not lie on a cycle of length g.

vuw is a 2-path in $G[C_X]$. Since $X_u^{+}(w) = X_u^{+}(v) = X_{vw}^{+}(u) = \varnothing$, $X_u^{=}(w) = (N(w) - u) \cap C_X, X_u^{=}(v) = (N(v) - u) \cap C_X, X_{vw}^{=}(u) = (N(u) - \{v, w\}) \cap C_X$. $N_{(g-4)/2}(X_u^{=}(v)) \cap N_{(g-4)/2}(X_u^{=}(w)) \cap X = \varnothing$ because u does not lie on a cycle of length g. $N_{(g-4)/2}(X_u^{=}(v)) \cap X, N_{(g-4)/2}(X_u^{=}(w)) \cap X, N_{(g-4)/2}(X_{vw}^{=}(u)) \cap X$ are pairwise disjoint because no cycle less than g exists. $N_{(g-4)/2}(u) \cap X, N_{(g-4)/2}(v) \cap X, N_{(g-4)/2}(w) \cap X, N_{(g-4)/2}(X_u^{=}(v) \cup X_u^{=}(w) \cup X_{vw}^{=}(u)) \cap X$ are also pairwise disjoint because no cycle less than g exists. So we deduce $\kappa_3(G) \geq \xi_3(G)$, a contradiction.

(4) Suppose that any vertex $u \in C_X$ satisfies $|N_{(g-4)/2}(u) \cap X| \geq 2$.

Take any 2-path uvw in $G[C_X]$. Notice that $X_v^{+}(u) = X_v^{+}(w) = X_{uw}^{+}(v) = \varnothing$. The hypothesis implies $|N_{(g-4)/2}(X_v^{=}(u)) \cap X| \geq 2|X_v^{=}(u)|, |N_{(g-4)/2}(X_v^{=}(w)) \cap X| \geq 2|X_v^{=}(w)|$, $|N_{(g-4)/2}(X_{uw}^{=}(v)) \cap X| \geq 2|X_{uw}^{=}(v)|, |N_{(g-4)/2}(u) \cap X| \geq |X_v^{-}(u)|, |N_{(g-4)/2}(w) \cap X| \geq |X_v^{-}(w)|$, $|N_{(g-4)/2}(v) \cap X| \geq |X_{uw}^{-}(v)|$. We can assume $|X_v^{=}(u)| \leq |X_v^{=}(w)|$. Notice that the sets $N_{(g-4)/2}(u) \cap X, N_{(g-4)/2}(w) \cap X, N_{(g-4)/2}(v) \cap X, N_{(g-4)/2}(X_{uw}^{=}(v)) \cap X$ and $N_{(g-4)/2}(X_v^{=}(w)) \cap X$ are pairwise disjoint, then we have

$$\begin{aligned}
\kappa_3(G) = |X| &\geq |N_{(g-4)/2}(u) \cap X| + |N_{(g-4)/2}(v) \cap X| \\
&\quad + |N_{(g-4)/2}(w) \cap X| + |N_{(g-4)/2}(X_v^{=}(w)) \cap X| + |N_{(g-4)/2}(X_{uw}^{=}(v)) \cap X| \\
&\geq |X_v^{-}(u)| + |X_{uw}^{-}(v)| + |X_v^{-}(w)| + 2|X_v^{=}(w)| + 2|X_{uw}^{=}(v)| \\
&\geq |X_v^{-}(u)| + |X_{uw}^{-}(v)| + |X_v^{-}(w)| + |X_v^{=}(w)| + |X_v^{=}(u)| + |X_{uw}^{=}(v)| \\
&= d(u) + d(v) + d(w) - 4 \geq \xi_3(G),
\end{aligned}$$